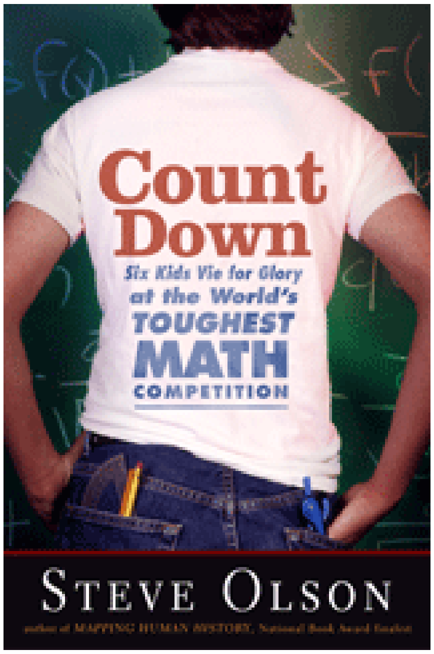


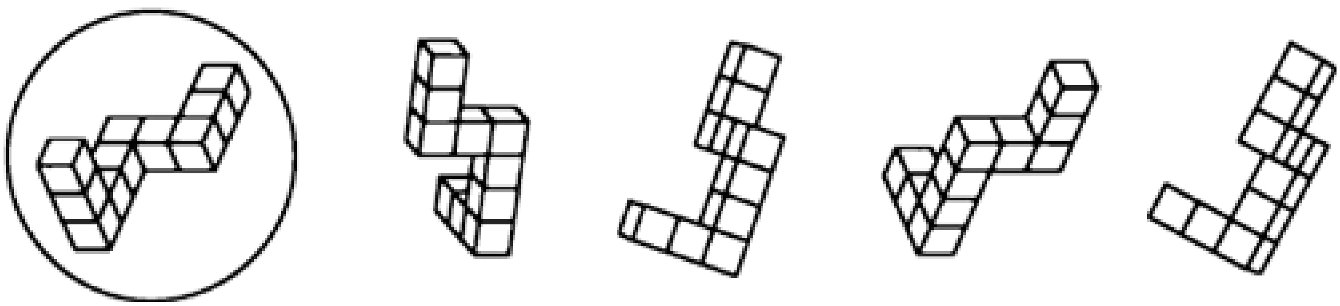
Count Down

by Steve Olson

- Cast of Characters
- A Conversation With Steve Olson
- *Count Down* Tour

Test your mathematical prowess

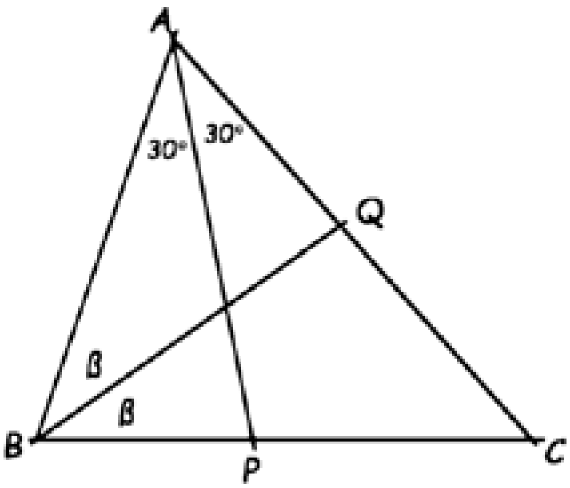
Q. Which of the following objects shown below are identical to the circled object?



Answer:

The first and fourth uncircled objects are identical to the circled object. This problem, which originally appeared in *Science* magazine in 1971, explored how people make these comparisons. Roger Shepard and his coauthor, Jacqueline Metzler, showed sets of three-dimensional block structures to people and asked them to determine whether the objects were the same. They found that the amount of time a person needed to answer the question depended on the extent to which an object had been rotated. Apparently, people were rotating the objects in their minds to see if they corresponded with the other objects. Shepard and Metzler were able to show that the average person could mentally rotate an object at about 60 degrees per second.

Q. Let ABC be a triangle with angle BAC= 60 degrees. Let AP bisect angle BAC and let BQ bisect angle ABC, with P on BC and Q on AC. If AB + BP = AQ +QB, what are the angles of the triangle?



Answer:

1. Extend line AB to a point called R.
2. Make BR equal in length to BP.
3. Since AB+BP = AQ+QB, you can prove that AR=AC.
4. Triangle BRP is an isosceles triangle with the big angle equal to 180 degrees —2 beta.
5. Thus, angle BRP has to equal beta.
6. If angle BRP=beta, then angle ACB also equals beta because line AP bisects angle A.
7. Therefore the two halves of the chevron formed by points A, R, P, and C are identical.
8. If angle C = beta and angle B=2 beta, then beta has to equal 40 degrees for the triangle ABC to add to 180 degrees.

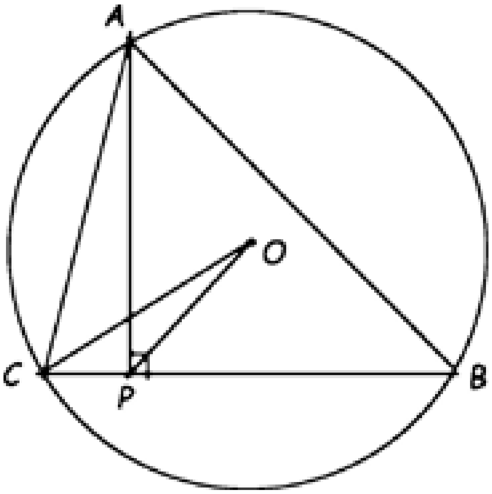
Q. Twenty-one girls and twenty-one boys took part in a mathematical competition. It turned out that (a) each contestant solved at most six problems, and (b) for each pair of a girl and a boy, there was at least one problem that was solved by both the girl and the boy. Prove that there was a problem that was solved by at least three girls and at least three boys.

	Boy 1	Boy 2	Boy 3	Boy 4	Boy 5
Girl 1	D	F	D	H	A
Girl 2	G	B	G	B	B
Girl 3	D	A	A	E	A

Answer:

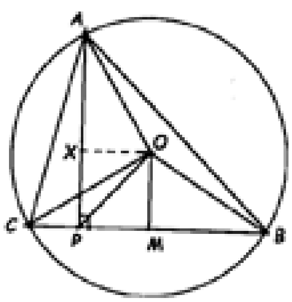
1. Draw a table listing the 21 boys in each column and the 21 girls in each row.
2. Your table will contain 21 x 21, or 441 boxes.
3. Mark a letter in each box representing a problem solved by both that girl and that boy.
4. The problem states that each girl-boy pair solved at least one problem in common, so all 441 boxes will have a letter.
5. Each contestant solved at most six problems, so only six different letters can appear in any given row or column of 21 boxes.
6. In general, six different letters can be placed in the 21 boxes in a row only if at least 11 of those boxes contain letters that appear three or more times in that row.
7. Go through each row and color red all the boxes containing letters that appear at least three times in that row.
8. Applying this rule to girl 3's problems would mean coloring red the A's in her row.
9. 11 boxes in each row must be colored red because of the restrictions on distributing 6 letters among 21 boxes.
10. Since the table has 21 rows, at least 11 x 21 or 231 boxes must be colored red.
11. Apply the same logic to the columns.
12. Using another color, say, blue, at least 231 of the boxes will be blue.
13. The table contains only 441 boxes and if at least 231 boxes are red and 231 boxes are blue, then some of the boxes will be both red and blue.
14. The letter in each of the doubly colored boxes represents a problem solved by at least three girls and at least three boys.

Q. In acute triangle ABC with circumcenter O and altitude AP, angle C is greater than or equal to angle B plus 30 degrees. Prove that angle A plus angle COP is less than 90 degrees.



Answer:

The following diagram shows the addition of lines from the circumcenter, O, perpendicular to Lines AP and CB. These perpendicular lines are labeled X and M, respectively. Now you should be able to solve the original problem.



1. Angle CAO—angle CAP=angle PAO.
2. You want to find angle CAO.
3. Angles AOC and ABC are related in a particular way. They share points C and A. Also point O is the center of the circumcircle and B is on the circumference of the circumcircle.
4. Therefore, according to a classic theorem in geometry, angle AOC is twice the measure of angle B.
5. Triangle CAO — is an isosceles triangle; two sides are the same length because they both are radii of the circle.
6. Therefore two of the angles are the same size.
7. Using algebra you can prove that angle CAO = 90 degrees—angle B.
8. You need to find the size of angle CAP.
9. One angle of the triangle formed by points C,A, and P is 90 degrees, and angle ACP is the same as angle C.
10. Therefore, angle CAP= 90 degrees —angle C.
11. Now you can calculate angle PAO: 90—C (the measure of angle CAP) from 90—B (the measure of CAO).
12. The two 90s cancel out.

13. Result: angle PAO=C—B.
14. To find angle COP, look at the distance from C to P and P to M.
15. Assume that P is closer to C than to M, then the distance CP is greater than PM.
16. Therefore, angle COP has to be smaller than angle OCP.
17. Angle OCP is part of the isosceles triangle OCB.
18. Like angle B, angle COB is equal to two times angle A.
19. Angle OCP is therefore equal to 90 degrees minus angle A.
20. Since angle COP is smaller than angle OCP, angle COP has to be less than 90 degrees minus angle A because that is the measure of angle OCP.

